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PREDICTION OF EFFECTIVE YIELDS OF DELAYED NEUTRONS IN MSRE

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ABSTRACT

Equations were developed and calculations were made to determine the effective contributions of delayed neutrons in the MSRE during steady power operation. Nonleakage probabilities were used as the measure of relative importance of prompt and delayed neutrons, and the spatial and energy distributions of the prompt and delayed neutron sources were included in the calculation of these probabilities.

Data which indicate a total yield of 0.0064 delayed neutron per neutron were used to compute total effective yields of 0.0067 and 0.0036 for the MSRE under static and circulating conditions respectively.

The effective fractions for the individual groups of delayed neutrons will be used in future digital calculations of MSRE kinetic behavior.

CONTENTS

INTRODUCTION

The kinetics of the fission chain reaction in a circulating-fuel reactor are influenced by the transport of the delayed neutron precursors. This fact makes a rigorous treatment quite complicated.[1,2] The approach generally followed is to drop the transport term from the precursor equation (making the kinetics equations identical with those for stationary reactors) and make an approximate allowance for the precursor transport by replacing the delayed neutron fractions, β_i, with "effective" values, β_i^*. This approximation is used in ZORCH, the computer program recently developed for analysis of the kinetics of the MSRE.[3]

The purpose of the work reported here is to obtain values for β_i^* to be used in the MSRE analyses.

APPROACH TO THE PROBLEM

The importance of delayed neutrons is enhanced in fixed-fuel reactors because the energy spectra of the delayed neutrons lie at much lower energies than that of the prompt neutrons. The differences in energy spectra make the delayed neutrons more valuable because they are less likely to escape from the reactor in the course of slowing down to thermal energies. This effect is, of course, also present in circulating-fuel reactors. Of greater importance, however, in these reactors is the <u>spatial</u> distribution of the delayed neutron sources. Many of the delayed neutrons are emitted outside the core and contribute nothing to the chain reaction. Furthermore, those delayed neutrons which are emitted in the core are, on the average, produced nearer to the edges of the core than are the prompt neutrons, which tends to further reduce the contribution of a particular group of delayed neutrons.

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[1] J. A. Fleck, Jr., Kinetics of Circulating Reactors at Low Power, <u>Nucleonics</u> **12**, No. 10, 52-55 (1954).

[2] B. Wolfe, Reactivity Effects Produced by Fluid Motion in a Reactor Core, <u>Nuclear Sci. and Eng.</u> **13**, 80-90 (1962).

[3] C. W. Nestor, Jr., ZORCH, an IBM-7090 Program for Analysis of Simulated MSRE Power Transients With a Simplified Space-Dependent Kinetics Model, ORNL TM -345 (Sept. 18, 1962).

The contribution of delayed neutrons during a power transient in a circulating-fuel reactor is affected by the continual change in the shape of the spatial source distribution. Thus the use of an "effective" fraction for a delay group in analyzing such transients is, in itself, an approximation. If this approximation is made, and a single set of "effective" fractions is to be used in the analysis of a variety of transients, it would seem that the values should be the fractional contributions of the various groups to the chain reaction under steady-state conditions. The problem at hand is to calculate these contributions.

It has been the practice in the analyses of circulating-fuel reactors to take β_i^*/β_i to be just the fraction of the ith group which is emitted inside the core. This implies that the importance of the delayed neutrons is equal to that of the prompt neutrons; which would be true if the increase in importance due to lower source energies exactly offset the decrease due to the distortion of the spatial distribution in the core. The further approximation is usually made in computing the fraction emitted in the core that the precursor production is uniform over the core volume. We shall seek to improve the evaluation of β_i^* by calculating more accurately the spatial distribution of precursors and by taking into account more explicitly the effect of the spatial and energy distributions on the importance of the delayed neutrons.

In a discussion of fixed-fuel reactors, Krasik quotes Hurwitz as defining β_i^*/β_i as "essentially the probability that a delayed neutron of the ith kind will produce a fission divided by the probability that a prompt neutron will produce a fission," and adds that "for a simple reactor this probability is given by the ratio of the nonleakage probabilities of the respective types of neutrons."[4] Let us adopt the definition of β_i^*/β_i as the ratio of the nonleakage probabilities. Suppose that the nonleakage probability for prompt neutrons is P_{pr} and for the delayed neutrons which are actually emitted in the core it is P_i. Of a particular group, only the fraction θ_i is emitted in the core so the nonleakage probability for all neutrons of the ith group is $\theta_i P_i$. Therefore

[4] S. Krasik, "Physics of Control," p 8-10 in Nuclear Engineering Handbook, ed. by H. Etherington, McGraw-Hill, New York, 1958.

$$\beta_i^* = \frac{\beta_i \theta_i P_i}{P_{pr}} \tag{1}$$

The nonleakage probabilities can be expressed in a simple form if the reactor is treated as a bare, homogeneous reactor. The source of prompt neutrons in the reactor is proportional to the fission rate, which follows closely the shape of the fundamental mode of the thermal neutron flux. In a homogeneous, cylindrical, bare reactor this is:

$$\phi(r,z) = \phi_o J_o(2.4\ r/R)\ \sin(\pi z/H) \tag{2}$$

For the prompt neutrons, with this spatial source distribution

$$P_{pr} = \frac{e^{-B^2\tau}}{1 + L^2B^2} \tag{3}$$

where

$$B^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \tag{4}$$

The spatial distributions of the delayed neutron sources are not the same as that of the prompt neutrons because of the transport of the precursors in the circulating fuel. The source distributions can be calculated from power distribution, fuel velocities, system volumes, etc. (This is done for a simple cylindrical reactor in the next section.) It is convenient, for the purpose of calculating leakage probabilities, to expand each source function in an infinite series:

$$S_i(r,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{imn}\ J_o(j_m r/R)\ \sin(n\pi z/H) \tag{5}$$

where j_m is a root of $J_o(x) = 0$. (See Appendix for derivation of nonleakage probabilities.) For neutrons with a source distribution

$$S_{mn} = A_{mn}\ J_o(j_m r/R)\ \sin(n\pi z/H) \tag{6}$$

the nonleakage probability is

$$P_{mn} = \frac{e^{-B_{mn}^2 \tau}}{1 + L^2 B_{mn}^2} \qquad (7)$$

where

$$B_{mn}^2 = \left(\frac{j_m}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2 \qquad (8)$$

For the ith group of delayed neutrons, then[5]

$$P_i = \frac{\sum_m \sum_n A_{imn} f \int_o^R \int_o^H J_o\left(\frac{j_m r}{R}\right) \sin\left(\frac{n\pi z}{H}\right) \frac{e^{-B_{mn}^2 \tau_i}}{1 + L^2 B_{mn}^2} \, 2\pi r \, dr \, dz}{f \int_o^R \int_o^H S_i(r,z) \, 2\pi r \, dr \, dz} \qquad (9)$$

Note that the age, τ_i, is the appropriate value for neutrons with the source energy distribution of group i.) If $S_i(r,z)$ is normalized to one fission neutron, then the denominator in (9) is $\beta_i \theta_i$. Thus, from (1), the numerator in (9) is just $\beta_i^* P_{pr}$. The numerator can be integrated to give

$$\beta_i^* P_{pr} = 4HR^2 f \sum_{m=1} \sum_{n=1,3..} A_{imn} \frac{J_1(j_m)}{n \, j_m} \frac{e^{-B_{mn}^2 \tau_i}}{1 + L^2 B_{mn}^2} \qquad (10)$$

(Only odd values of n remain in the summation because the contribution of all even values of n to the integral is zero.)

The approach we shall follow in calculating effective delayed neutron fractions is then as follows: calculate the steady-state source distributions, $S_i(r,z)$, in a bare, homogenized approximation of the MSRE core; evaluate the coefficients, A_{imn}; compute β_i^* from (10). By "bare, homogenized approximation" we mean a reactor in which the flux is assumed to vanish at the physical boundary and in which the composition is uniform

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[5]It is assumed here that the fuel volume fraction, f, is not a function of r. This is also implied in (2) and (3).

so that (2) applies.[6] We shall also assume that the fuel velocity is uniform over the entire core.

DERIVATION OF EQUATIONS

Steady-State Concentrations of Precursors

Let us derive the formula for the steady-state concentration of the precursor of a group of delayed neutrons as a function of position in the core.

Begin by considering an elemental volume of fuel as it moves up through a channel in the core. The precursor concentration in the fuel as it moves along is governed by

$$\frac{dc}{dt} = \beta\nu\Sigma_f\phi(r,z) - \lambda c \tag{11}$$

The fuel rises through the channel with a constant velocity v so

$$\frac{dc}{dz} = \frac{dc}{dt}\bigg/\frac{dz}{dt} = \frac{1}{v}\,\frac{dc}{dt} \tag{12}$$

With the substitution of (2) and (12), equation (11) becomes

$$\frac{dc}{dz} = \frac{\beta\nu\Sigma_f\phi_o}{v} J_o\left(\frac{2.4r}{R}\right)\sin\frac{\pi z}{H} - \frac{\lambda}{v}c \tag{13}$$

Along any channel r is constant and at steady state when ϕ_o is not changing, (13) can be integrated to give

$$c(r,z) = \frac{\beta\nu\Sigma_f\phi_o\lambda J_o(\frac{2.4r}{R})}{\lambda^2 + (\frac{\pi v}{H})^2}\left[\sin\frac{\pi z}{H} - \frac{\pi v}{\lambda H}\left(\cos\frac{\pi z}{H} - e^{-\lambda z/v}\right)\right] + c_o e^{-\lambda z/v} \tag{14}$$

where c_o is the concentration in fuel entering the core at z = 0.

The concentration at the outlet of a channel is given by (14) with z = H.

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[6] In the actual MSRE core, the flux deviates from (2) because of the depression around the rod thimbles and because the flux does not vanish at the physical edge of the core.

$$c(r,H) = \frac{\beta\nu\Sigma_f\phi_o(1 + e^{-\lambda H/v})\, J_o(\frac{2.4r}{R})}{\left[1 + \left(\frac{\lambda H}{\pi v}\right)^2\right]\frac{\pi v}{H}} + c_o e^{-\lambda H/v} \qquad (15)$$

The concentration of precursors in the fuel leaving the core is the mixed mean of the streams from all of the channels.

$$c_1 = \frac{\int_o^R c(r,H)v(r)f(r)\, 2\pi r\, dr}{\int_o^R v(r)f(r)\, 2\pi r\, dr} \qquad (16)$$

We have assumed that f and v are constant across the core. With this assumption, substitute (15) in (16) and integrate to obtain

$$c_1 = c_o e^{-\lambda t_c} + \frac{2\beta\nu\Sigma_f\phi_o t_c(1 + e^{-\lambda t_c})\, J_1(2.405)}{2.405\,\pi\left[1 + \left(\frac{\lambda t_c}{\pi}\right)^2\right]} \qquad (17)$$

where t_c has been substituted for H/v. The precursors decay during the time t_x required for the fuel to pass through the external loop. Thus

$$c_o = c_1 e^{-\lambda t_x} \qquad (18)$$

Equations (17) and (18) can be solved for c_o.

$$c_o = \frac{2\beta\nu\Sigma_f\phi_o t_c(1 + e^{-\lambda t_c})\, e^{-\lambda t_x}\, J_1(2.405)}{2.405\pi\left[1 + \left(\frac{\lambda t_c}{\pi}\right)^2\right]\left[1 - e^{-\lambda(t_c+t_x)}\right]} \qquad (19)$$

When this is substituted into (14) the desired expression for c(r,z) is obtained.

Normalized Source Distributions

As explained on page 6 it is desirable to normalize the delayed neutron source to one fission neutron so that the integral over the core volume will equal $\beta\theta$. The rate of neutron production is

$$N = \int_{V_{fc}} \nu\Sigma_f \phi dV_{fc} = \nu\Sigma_f \phi_o \int_0^R \int_0^H J_o\left(\frac{2.4r}{R}\right) \sin\left(\frac{\pi z}{H}\right) f\, 2\pi r\, dr\, dz \tag{20}$$

Assuming, as before, that the fuel volume fraction is not a function of r, integration gives

$$N = 4HR^2 f\, \nu\Sigma_f \phi_o\, J_1(2.405)/2.405 \tag{21}$$

for the total rate of neutron production. The normalized source distribution is

$$S_i(r,z) = \lambda_i c_i(r,z)/N \tag{22}$$

Substitution of (14), (19) and (21) into (22) gives for each group

$$S(r,z) = s_o e^{-\lambda t_c z/H} + \left[s_1 \sin\frac{\pi z}{H} - s_2 \cos\frac{\pi z}{H} + s_2 e^{-\lambda t_c z/H}\right] J_o\left(\frac{2.405\, r}{R}\right) \tag{23}$$

where

$$s_o = \beta \frac{(1 + e^{-\lambda t_c})\, e^{-\lambda t_x}\, (\lambda t_c/\pi)}{2\, HR^2 f\left[1 + \left(\frac{\lambda t_c}{\pi}\right)^2\right]\left[1 - e^{-\lambda(t_c + t_x)}\right]} \tag{24}$$

$$s_1 = \beta \frac{2.405\, (\lambda t_c/\pi)^2}{4\, HR^2 f\left[1 + \left(\frac{\lambda t_c}{\pi}\right)^2\right] J_1(2.405)} \tag{25}$$

$$s_2 = \beta \frac{2.405\ (\lambda t_c/\pi)}{4\ HR^2 f \left[1 + \left(\frac{\lambda t_c}{\pi}\right)^2\right] J_1(2.405)} \tag{26}$$

Expansion of Sources in Series Form

For the purpose of calculating the nonleakage probability let us represent $S(r,z)$ by an infinite series which vanishes at $z = 0$, $z = H$ and $r = R$.

$$S(r,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\ J_o(\alpha_m r)\ \sin(\gamma_n z) \tag{27}$$

The condition that $S(r,z)$ vanish at $r = R$ is satisfied if $\alpha_m = j_m/R$ where j_m is the mth root of $J_o(x) = o$. The boundary conditions at the ends are satisfied by $\gamma_n = n\pi/L$.

Expansion of the functions in (23) gives[7]

$$S_c e^{-\lambda t_c Z/L} = \frac{4S_o}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{j_m J_1(j_m)} \left[\frac{1 + (-1)^{n+1} e^{-\lambda t_c}}{n^2 + \left(\frac{\lambda t_c}{\pi}\right)^2}\right] J_o\left(\frac{J_m r}{R}\right) \sin\left(\frac{n\pi z}{H}\right) \tag{28}$$

$$-S_2\ J_o\left(\frac{2.405\ r}{R}\right) \cos\frac{\pi z}{H} = -\ \frac{2\ S_2}{\pi} J_o\left(\frac{2.405\ r}{R}\right) \sum_{n=2}^{\infty}\left[\frac{1 + (-1)^n}{n^2 - 1}\right] n \sin\frac{n\pi z}{H} \tag{29}$$

$$S_2 J_o\left(\frac{2.405\ r}{R}\right) e^{-\lambda t_c z/H} = \frac{2\ S_2}{\pi} J_o\left(\frac{2.405\ r}{R}\right) \sum_{n=1}^{\infty} \left[\frac{1 + (-1)^{n+1}\ e^{-\lambda t_c}}{n^2 + \left(\frac{\lambda t_c}{\pi}\right)^2}\right] n \sin\frac{n\pi z}{H} \tag{30}$$

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[7] For a discussion of expansion in series of Bessel functions and half-range sine series expansions see, for instance, C. R. Wylie, Jr., Advanced Engineering Mathematics, 2d ed., p 432-37 and 253-57, McGraw-Hill, New York, 1960.

Thus for m = 1 and n = 1

$$A_{11} = \frac{2}{\pi}\left[S_2 + \frac{2\,S_o}{2.405\,J_1(2.405)}\right]\frac{1 + e^{-\lambda t_c}}{1 + \left(\frac{\lambda t_c}{\pi}\right)} + S_1 \tag{31}$$

For m = 1 and all n > 1

$$A_{1n} = \frac{2n}{\pi}\left\{\left[S_2 + \frac{2\,S_o}{2.405\,J_1(2.405)}\right]\left[\frac{1 + (-1)^{n+1}e^{-\lambda t_c}}{n^2 + \left(\frac{\lambda t_c}{\pi}\right)^2}\right] - S_2\left[\frac{1 + (-1)^n}{n^2 - 1}\right]\right\} \tag{32}$$

For m > 1 and all n

$$A_{mn} = \frac{4\,S_o n}{\pi j_m J_1(j_m)}\left[\frac{1 + (-1)^{n+1}\,e^{-\lambda t_c}}{n^2 + \left(\frac{\lambda t_c}{\pi}\right)^2}\right] \tag{33}$$

Fraction Emitted in the Core

The equations derived in the foregoing sections permit the evaluation of β_i^* without the explicit calculation of θ_i. Further insight may be obtained by calculating θ_i, and this can be done most easily by using the relation

$$\theta = 1 - \frac{Q(c_1 - c_o)}{\beta N} \tag{34}$$

where Q is the volumetric circulation rate of the fuel. This is given by

$$Q = \pi R^2 fv \tag{35}$$

When (35), (21), (19), and (18) are substituted in (34), the result simplifies to

$$\theta_i = 1 - \frac{1}{2}\left[\frac{1}{1 + \left(\frac{\lambda_i t_c}{\pi}\right)^2}\right]\left[\frac{(1 + e^{-\lambda_i t_c})(1 - e^{-\lambda_i t_x})}{1 - e^{-\lambda_i (t_c + t_x)}}\right] \tag{36}$$

It is of interest to compare this relation, which takes into account the spatial distribution of the precursor production, with the relation obtained[8] when the precursor production is assumed to be flat over the core volume. The latter relation is

$$\zeta_i = 1 - \frac{1}{\lambda_i t_c}\left[\frac{(1 - e^{-\lambda_i t_c})(1 - e^{-\lambda_i t_x})}{1 - e^{-\lambda_i (t_c + t_x)}}\right] \qquad (37)$$

The digital programs for MSRE kinetics calculations (MURGATROYD[9] and ZORCH[10]) have as an integral part the computation of delayed neutron fractions from precursor yields and decay constants and the reactor residence times, all of which are input numbers. The fraction computed and used for each group is $\beta_i \zeta_i$ where ζ_i is given by equation (37). Therefore, in order to have the kinetics calculations done with β_i^* for the fractions, it is necessary to put in a fictitious value of β_i, equal to β_i^*/ζ_i.

Effective Fraction in Noncirculating System

The change in the effective delayed neutron fraction between noncirculating and circulating conditions is a factor in determining control rod requirements.

In the noncirculating core, the source of delayed neutrons has the same shape as the source of prompt neutrons and

$$\beta_{is}^* = \beta_i \frac{e^{-B_{11}^2 \tau_i}}{e^{-B_{11}^2 \tau_{pr}}} \qquad (38)$$

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[8] P. R. Kasten, Dynamics of the Homogeneous Reactor Test, ORNL-2072 (June 7, 1956).

[9] C. W. Nestor, Jr., MURGATROYD, An IBM-7090 Program for the Analysis of the Kinetics of the MSRE, ORNL-TM-203 (Apr. 6, 1962).

[10] C. W. Nestor, Jr., ZORCH, an IBM-7090 Program for Analysis of Simulated MSRE Power Transients with a Simplified Space-Dependent Kinetics Model, ORNL-TM-345 (Sept. 18, 1962).

It may also be of interest to compare the magnitudes of the delayed neutron source distributions under static and circulating systems. In the static system

$$S_{is}(r,z) = \frac{\beta_i \nu\Sigma_f \phi_o J_o(\frac{2.4\ r}{R})\ \sin(\frac{\pi}{H})}{N} \tag{39}$$

N is given by (21), and (39) reduces to

$$S_{is}(r,z) = \frac{\beta_i\ 2.405}{4HR^2 f\ J_1(2.405)} J_o\ (\frac{2.4\ r}{R})\ \sin(\frac{\pi z}{H}) \tag{40}$$

The same result is obtained if one substitutes $t_c = t_x = \infty$ in equations 23-26.

RESULTS OF MSRE CALCULATIONS

The equations derived in the preceding section were used in calculations for a simplified model of the MSRE core. (See Appendix for data used.) Results are summarized in Table 1. The table shows that the core residence time, in units of precursor half-lives, ranges from 0.2 for the longest-lived group to 41 for the shortest-lived group. Because of this wide range, the shapes of the delayed neutron sources vary widely. Figure 1 shows axial distributions at the radius where $J_o(2.4\ r/R)$ has its average value, 0.4318. The source densities were normalized to a production of one fission neutron in the reactor. For the longest-lived group, the S_o term, which is flat in the radial direction, is by far the largest. This term is relatively insignificant for groups 3-6. For the very short-lived groups, the S_1 term predominates, i.e., the shape approaches that of the fission distribution.

Figure 2 shows the twofold effect of circulation in reducing the contribution of the largest group of delayed neutrons. The reduction in the number of neutrons emitted in the core is indicated by the difference in the areas under the curves. The higher leakage probability with the fuel circulating is suggested by the shift in the distribution, which reduces the average distance the neutrons would travel in reaching the outside of the core.

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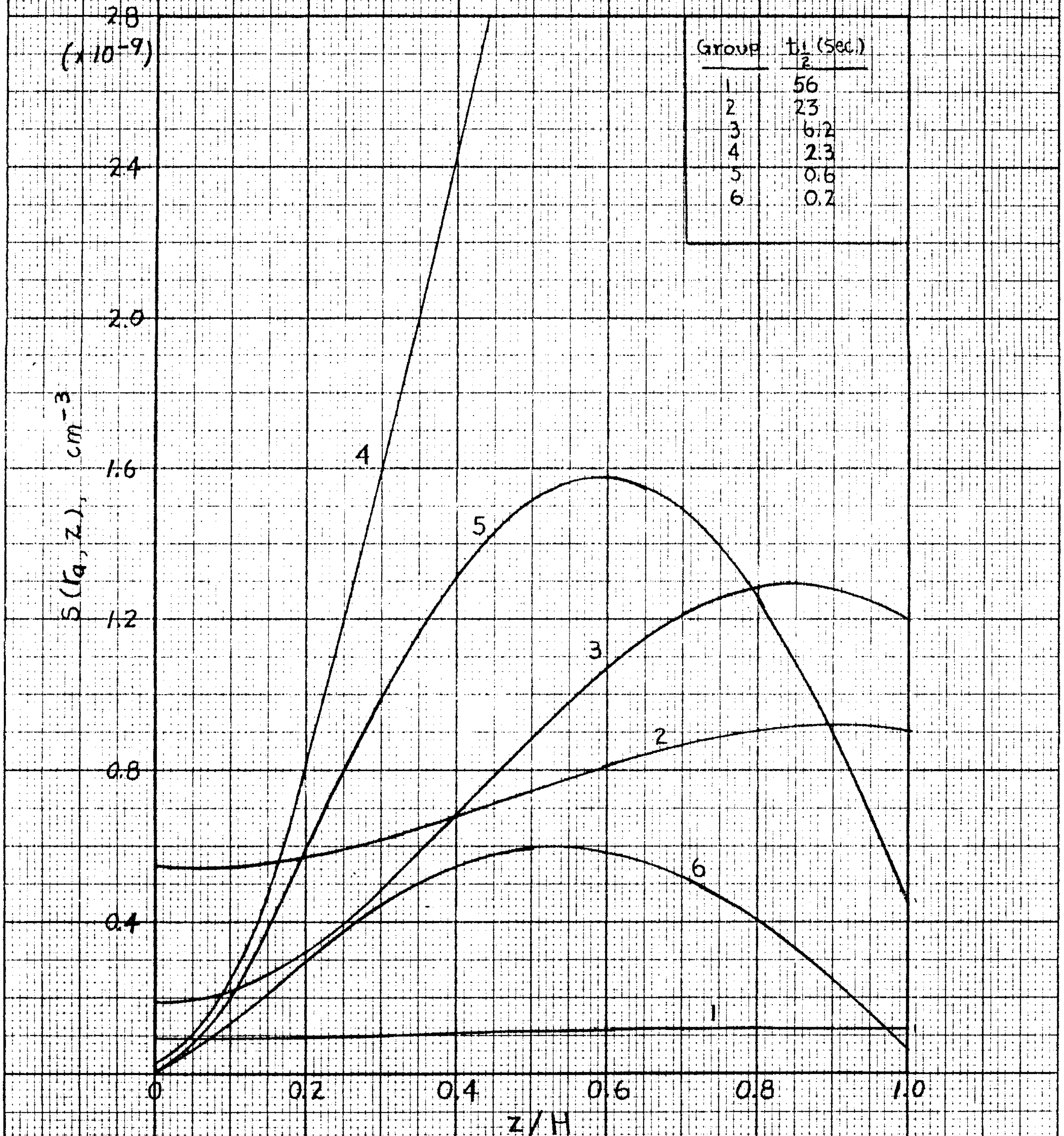

Fig. 1 Delayed Neutron Source Distributions in MSRE Core With Fuel Circulating at 1200 GPM.

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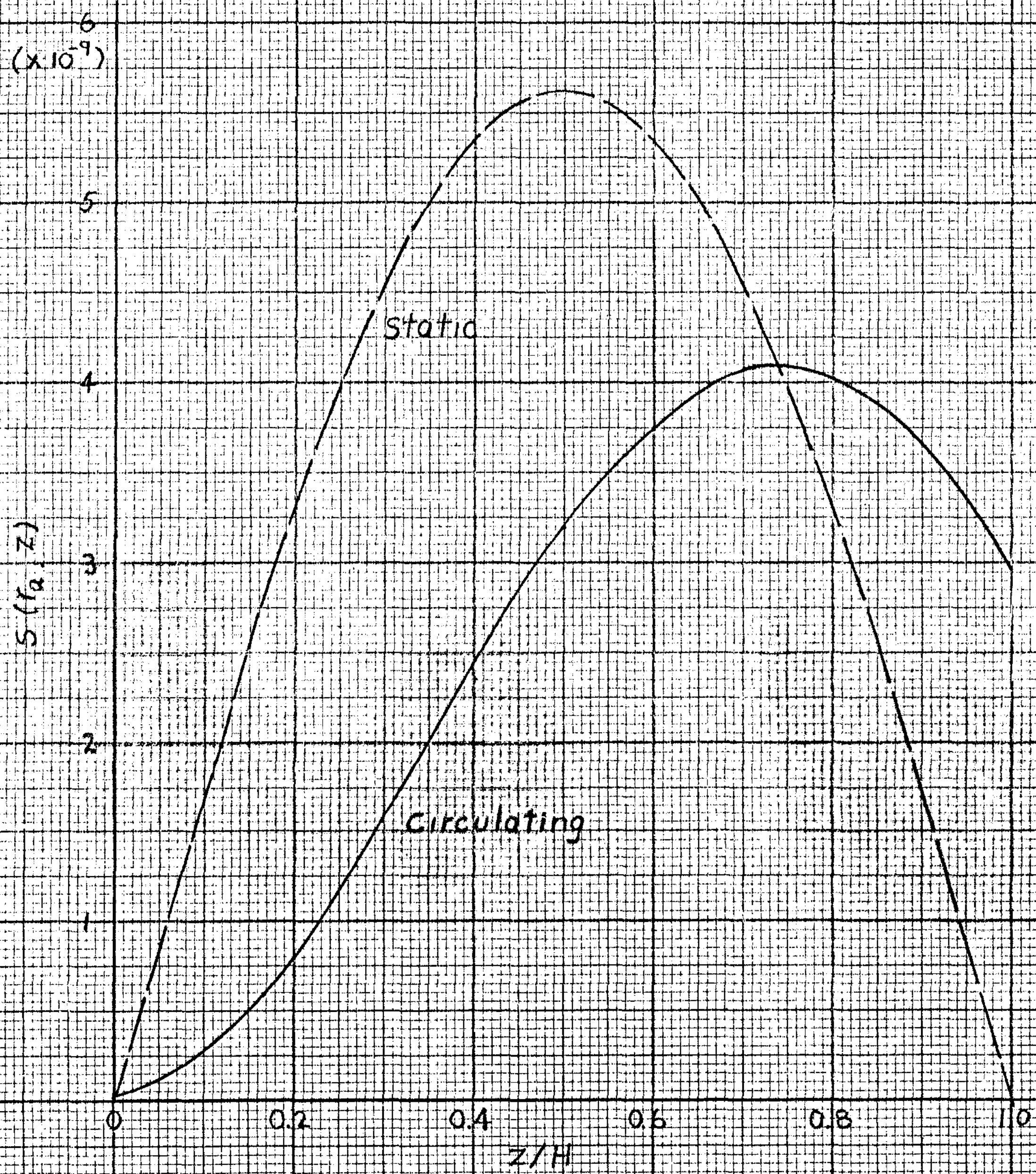

Fig 2. Comparison of Source Distributions for 23-sec. Group in Static and in Circulating MSRE Core.

Table 1. Delayed Neutrons in MSRE

Group	1	2	3	4	5	6
$t_{\frac{1}{2}}$ (sec)	55.7	22.7	6.22	2.30	0.61	0.23
$t_c/t_{\frac{1}{2}}$	0.17	0.41	1.51	4.07	15	41
$t_x/t_{\frac{1}{2}}$	0.30	0.72	2.64	7.15	27	72
S_o/S_1	65.7	9.24	0.30	3.6×10^{-3}	9×10^{-10}	2×10^{-23}
S_2/S_1	27.0	10.99	3.01	1.11	0.29	0.11
θ_i	0.364	0.371	0.458	0.709	0.960	0.994
P_i/P_{pr}	0.676	0.718	0.868	0.906	1.010	1.031
β_i^*/β_i	0.246	0.266	0.398	0.672	0.970	1.025
ζ_i	0.364	0.370	0.448	0.669	0.906	0.965
P_{is}/P_{pr}	1.055	1.039	1.043	0.948	1.010	1.031
$10^4\ \beta_i$	2.11	14.02	12.54	25.28	7.40	2.70
$10^4\ \beta_i^*$	0.52	3.73	4.99	16.98	7.18	2.77
$10^4\ \beta_{is}^*$	2.23	14.57	13.07	26.28	7.66	2.80

Totals for all six groups taken from Table 1 are listed in Table 2 for ease of comparison.

Table 2. Total Delayed Neutron Fractions in MSRE

Actual yield, $\sum \beta_i$	0.006405
Effective fraction in static system, $\sum \beta^*_{is}$	0.006661
Fraction emitted in core, circulating, $\sum \beta_i \theta_i$	0.003942
Effective fraction, circulating, $\sum \beta^*_i$	0.003617

APPENDIX

Expressions for Nonleakage Probabilities

It is desired to calculate the probability that a neutron from a distributed source of specified shape and initial energy will be absorbed as a slow neutron in a cylindrical, bare reactor.

Consider the neutrons to be born at energy E_{io} with a spatial distribution $S_i(r,z)$. Use age treatment to describe slowing down, i.e.,

$$\nabla^2 q\ (r,z,\tau) = \frac{\partial q}{\partial \tau}\ (r,z,\tau) \qquad (a1)$$

$$q(r,z,\tau_{io}) = S_i(r,z) \qquad (a2)$$

The steady-state equation for the thermalized neutrons is

$$D\ \nabla^2 \phi_i(r,z) - \Sigma_a \phi_i(r,z) + q_i(r,z,\tau_{th}) = 0 \qquad (a3)$$

The probability to be calculated is

$$P_i = \frac{\int_0^R \int_0^H \Sigma_a \phi_i(r,z)\ 2\pi r\ dr\ dz}{\int_0^R \int_0^H S_i(r,z)\ 2\pi r\ dr\ dz} \qquad (a4)$$

If (a1) is solved with the condition that q vanish at the boundaries and be finite everywhere inside the reactor, the solution is

$$q_i = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{imn}\ J_o\left(\frac{J_m r}{R}\right) \sin\left(\frac{n\pi z}{H}\right) e^{-B^2_{mn}(\tau - \tau_{io})} \qquad (a5)$$

where

$$B^2_{mn} = \left(\frac{jm}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2 \qquad (a6)$$

It is possible *a priori* to represent $S_i(r,z)$ and $\phi_i(r,z)$ by series

$$S_i(r,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{imn}\, J_o\left(\frac{J_m r}{R}\right)\sin\left(\frac{n\pi z}{H}\right) \tag{a7}$$

$$\phi_i(r,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{imn}\, J_o\left(\frac{J_m r}{R}\right)\sin\left(\frac{n\pi z}{H}\right) \tag{a8}$$

From (a2), (a5), and (a7) it is evident that

$$a_{imn} = A_{imn} \tag{a9}$$

And if ϕ is represented by (a8),

$$D\,\nabla^2\phi = -\,D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^2\, F_{imn}\, J_o\left(\frac{J_m r}{R}\right)\sin\left(\frac{n\pi z}{H}\right) \tag{a10}$$

Substitute (a5), (a9) and (a10) into (a3). Because of the orthogonality of the functions, one has for each mn

$$-\,DB_{mn}^2 F_{imn} - \Sigma_a F_{imn} + A_{imn}\, e^{-B_{mn}^2 \tau_i} = 0 \tag{a11}$$

where, for convenience, $\tau_{th} - \tau_{io}$ is represented by τ_i. From (a11)

$$F_{imn} = \frac{A_{imn}}{\Sigma_a}\,\frac{e^{-B_{mn}^2\tau_i}}{1 + L^2 B_{mn}^2} \tag{a12}$$

Now one can write

$$P_i = \frac{\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{imn}\left(\frac{e^{-B_{mn}^2\tau_i}}{1 + L^2 B_{mn}^2}\right)\int_0^R\int_0^H J_o\left(\frac{J_m r}{R}\right)\sin\left(\frac{n\pi z}{H}\right)2\pi r\,dr\,dz}{\displaystyle\int_0^R\int_0^H S_i(r,z)\,2\pi r\,dr\,dz} \tag{a13}$$

This is the expression for the nonleakage probability given in the text as equation (9).

Now consider a special case where the neutron source is proportional to the flux, namely the fission neutrons born at the site of the fission.

$$S_i(r,z) = k\Sigma_a \phi_i(r,z) \qquad \text{(a14)}$$

If k is not a function of position, insertion of (a14) into (a4) gives

$$P_i = \frac{1}{k} \qquad \text{(a15)}$$

Let us find an expression for k as follows: Substitute (a7) and (a8) for S_i and ϕ_i in (a14). This results in the relation

$$A_{imn} = k\Sigma_a F_{imn} \qquad \text{(a16)}$$

Substitute this in (a12)

$$F_{imn} = F_{imn} \frac{k\, e^{-B_{mn}^2 \tau_i}}{1 + L^2 B_{mn}^2} \qquad \text{(a17)}$$

This is satisfied for all mn for which $F_{imn} = 0$. If $F_{imn} \neq 0$, it must be that

$$\frac{k\, e^{-B_{mn}^2 \tau_i}}{1 + L^2 B_{mn}^2} = 1 \qquad \text{(a18)}$$

Because B_{mn}^2 is, in general, different for each mn, (a18) can only be true for one mn and therefore F_{imn} must be zero for all but that mn. It can be shown by consideration of the time-dependent neutron equations, that in a reactor free of extraneous sources, the steady-state flux corresponds to the fundamental mode, i.e., m = n = 1. (See Glasstone and Edlund, 12.37 - 12.41.) Therefore

$$P_i = \frac{1}{k} = \frac{e^{-B_{11}^2 \tau_i}}{1 + L^2 B_{11}^2} \qquad \text{(a19)}$$

This is the expression used in the text as the nonleakage probability for the prompt neutrons.

Data for MSRE Calculations

This section presents the data which were used in calculations for the MSRE. It includes dimensions and properties of the reactor and data on delayed neutrons.

MSRE Dimensions

Reactor dimensions which are required are H, R, f, v (or t_c) and t_x.

For R let us use 27.75 in., the inside radius of the INOR-8 can around the core.

Assigning values to H and v is not simple, because the fission distribution extends past the limits of the graphite core into the upper and lower heads. The axial distribution of the fission rate closely follows $\sin(\pi z/H)$ where H is 77.7 in. The longest graphite stringers are 68.9 in. long, and the channel region is only 62 in. Further complicating the situation is the fact that outside of the channeled region, the fuel velocity is lower than in the channels (because the volume fraction of fuel is much higher in the end regions). There are also radial variations in the fuel volume fraction and channel velocity. (In the central channels the fuel velocity is over three times the 0.60 ft/sec which is found in more than three-fourths of the channels.) Let us use the following simplifications. Consider the "core" to be bounded by the horizontal planes at the extreme top and the extreme bottom of the graphite. This gives H = 68.9 in. Enclosed by these boundaries is a total volume of 96.4 ft^3, of which 25.0 ft^3 is occupied by fuel. Thus f = 0.259. The residence time of fuel in the "core," at a flow rate of 1200 gpm, is 9.37 sec. The velocity, H/t_c, is 0.61 ft/sec. The total volume of circulating fuel is 69.1 ft^3, for a total circuit time of 25.82 sec. Thus t_x = 25.82 - 9.37 = 16.45 sec.

Precursor Yields and Half-Lives

For yields and half-lives of the delayed neutron groups, let us use the data of Keepin, Wimett and Zeigler for fission of U^{235} by thermal neutrons.[11] These are given in Table A-1.

[11] S. Krasik, "Physics of Control," p. 8-4 in Nuclear Engineering Handbook, ed. by H. Etherington, McGraw-Hill, New York, 1958.

Table A-1. Precursor Data

Group	1	2	3	4	5	6
Half-life (sec)	55.7	22.7	6.22	2.30	0.61	0.23
Decay constant, λ_i (sec^{-1})	0.0124	0.0305	0.1114	0.3013	1.140	3.010
Fractional yield, $10^4\beta_i$, ($n/10^4 n$)	2.11	14.02	12.54	25.28	7.40	2.70

Neutron Energies and Ages

The age of neutrons is given by

$$\tau = \int_{E}^{E_o} \frac{D}{\xi\Sigma_s} \frac{dE}{E} \tag{a20}$$

The age of prompt neutrons, which have an initial mean energy of about 2 Mev, is about 292 cm^2 in the MSRE core at 1200°F.[12] Let us estimate the age to thermal energy of the delayed neutrons as follows.

$$\tau = \left(\frac{D}{\xi\Sigma_s}\right)_{av} \int_{E_{th}}^{E_o} \frac{dE}{E} = \left(\frac{D}{\xi\Sigma_s}\right)_{av} \ell n \frac{E_o}{E_{th}} \tag{a21}$$

Therefore use the approximation that

$$\tau_i \approx \frac{\ell n \,(E_i/E_{th})}{\ell n \,(E_{pr}/E_{th})} \tau_{pr} \tag{a22}$$

The average energy of thermal neutrons at 1200°F is 0.119 ev.

Goldstein[13] reviewed the data on delayed neutron energies and recommended values for the first five groups. His values are given in Table A-2, together with values of τ_i calculated from (a22). No experimental values for the mean energy of the shortest-lived group are available, so a value of 0.5 Mev was assumed.

Table A-2. Neutron Energies and Ages to Thermal in MSRE Core

Group	1	2	3	4	5	6	Prompt
Mean Energy (Mev)	0.25	0.46	0.40	0.45	0.52	0.5	2.0
Age, τ_i (cm^2)	256	266	264	266	269	268	292

[12] This number is the ratio D/Σ for fast neutrons calculated by MODRIC, a multigroup diffusion calculation.

[13] H. Goldstein, Fundamental Aspects of Reactor Shielding, p 52, Addison-Wesley, Reading, Mass. (1959).

Neutron Diffusion Length

In the main body of the MSRE core the square of the diffusion length for thermal neutrons is 210 cm^2. This is for the core at 1200°F, containing fuel with no thorium and about 0.15 mole percent uranium.

Nomenclature

A_{imn}	coefficients in series expansion of S_i
a_{imn}	coefficients in series expansion of q_i
B^2	geometric buckling
c	precursor concentration in fuel
c_o	c entering core
c_1	mixed mean c leaving core
D	neutron diffusion coefficient
E	neutron energy
E_o	mean initial E
E_{th}	thermal E
F_{imn}	coefficient in series expansion of flux
f	volume fraction of fuel in core
H	height of cylindrical core
J_o, J_1	Bessel functions of the first kind
j_m	mth root of $J_o(x) = 0$
k	neutron multiplication factor
L	neutron diffusion length
N	total rate of neutron production in reactor
P	nonleakage probability
Q	volumetric circulation rate of fuel through core
q	neutron slowing-down density

Nomenclature - cont'd

r	radial distance from core axis
r_a	value of r where $J_o(2.4r/R) = 0.432$.
R	outside radius of cylindrical core
S	neutron source per unit volume of fuel
t_c	residence time of fuel in core
t_x	residence time of fuel in external loop
$t_{\frac{1}{2}}$	half-life of precursors
v	fuel velocity in core
z	axial distance from bottom of core
β_i	fraction of neutrons which belongs to ith group
β_i^*	"effective" fraction
ζ_i	calculated fraction of neutrons of group i which is emitted in core, assuming flat production of precursors
θ_i	fraction of neutrons of group i which is emitted in core
λ	precursor decay constant
ν	total neutrons produced per fission
ξ	average decrement in log E per collision
Σ_a	neutron absorption cross-section
τ	neutron age
ϕ	neutron flux

Subscripts

i	a group of neutrons
pr	prompt neutrons
s	noncirculating system

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